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Anamet Report No. 86.016
February 27, 1987

RAPID THERMAL LOADING OF DELAMINATED COMPOSITE STRUCTURES

PHASE I FINAL REPORT

Solicitation No. 86.1
Topic No. AF86-132
Contract No. F33615-86-C-3218

For

AFWAL/GLXPF
Area B, Building 45, Room 149
Wright-Patterson AFB, Ohio 45433

By

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Anamet Laboratories, Inc.

HAYWARD, CALIFORNIA

Anamet Report No. 86.016

February 27, 1987

This report documents the development of theory for the rapid thermal loading of delaminated composites performed under Air Force Contract No. F33615-86-C-3218 for AFWAL/GLXPF. The cognizant Air Force project engineer was Ms. Marge Artley. The principal investigator was Dr. Rocky R. Arnold.

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Vice President

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LIST OF ILLUSTRATIONS

NOMENCLATURE

a_{ij}, b_{ij}	Arbitrary interlaminar shear coefficients
$\vec{a}, \vec{b}$	Vectors of arbitrary interlaminar shear coefficients
b	Plate width
E_{ℓ_k}, E_{t_k}	Longitudinal and transverse Young's moduli of k'th lamina, respectively, with respect to material axes
e	Subscript denoting linear-elastic-material component
$\vec{e}, \vec{f}, \vec{g}$	Vectors of arbitrary displacement coefficients
F'	Stress-energy-density function
$F'_{fm_{e_k}}, F'_{fm_{p_k}}$, F'_{fm_k}	Elastic, elasto-plastic, and total stress-energy-density functions for the k'th lamina as a function of lamina inplane stresses, respectively
$F'_{m_{e_k}}, F'_{m_{p_k}}$, F'_{m_k}	Elastic, elasto-plastic, and total stress-energy-density functions for the matrix material between k'th and (k+1)st lamina as a function of transverse stresses acting between those two layers, respectively
G_m	Matrix material shear modulus
$G_{\ell t_k}$	k'th lamina inplane shear modulus with respect to lamina material axes
h	Laminate thickness
$\vec{h}_p (p=1,2,3)$	Vectors of arbitrary stress coefficients
$\vec{1}_N$	N-dimensional unit vector
$K_{\ell_k}, K_{t_k}, K_{\ell t_k}$	Ramberg-Osgood type constants for k'th lamina relative to the material axes
k	Index representing the k'th lamina

L	Plate length
N	Total number of laminae in the laminate
N_x, N_y, N_{xy}	Laminate inplane stress resultants
$n_{\ell_k}, n_{t_k}, n_{\ell t_k}$	Ramberg-Osgood type exponents for the k'th lamina relative to the material axes
$\vec{N}$	Laminate membrane stress resultant vector
p	Subscript denoting nonlinear-elastic-material component
p_{i_k} $(i=1,2,3)$	Exponents analogous to Ramberg-Osgood-type exponents defined in Reference 17
p_4	Ramberg-Osgood-type exponent, defined in Reference 17
q	Heat flux
T	Superscript denoting matrix transpose
t	Laminate thickness
t_{fm_k}	k'th lamina thickness over which the extensional stresses σ_{x_k} and σ_{y_k} and the shearing stress τ_{xy_k} act
t_{m_k}	Thickness of matrix material between the k'th and (k+1)st lamina's median-surfaces over which the interlaminar shear stresses τ_{zx_k} and τ_{yz_k} act
U''	Modified-Reissner functional
u_k, v_k	Median-surface x and y displacement functions of the k'th lamina, respectively
$\vec{u}, \vec{v}$	N-dimensional x and y displacement vectors, respectively
V	Volume
w	Lateral displacement funciton of laminate
x, y, z	Plate coordinates

$\gamma_{xy}^{(0)}, \gamma_{xy}^{(1)} \ldots$	Laminate averaged inplane shear stresses relative to plate axes
γ_{xy_k}	Median-surface shearing strains of k'th lamina relative to plate axes
$\gamma_{zx_k}, \gamma_{yz_k}$	Interlaminar shearing strains acting between the median-surfaces of the k'th and (k+1)st laminae relative to plate axes
ε_{eff_k}	Effective strain defined in Equation (18)
$\eta_{12,2}, \eta_{12,1}$	Lamina coefficients of mutual influence relative to material axes
$\eta_{t,\ell t}, \eta_{\ell,\ell t}$ $\eta_{\ell t,t}, \eta_{\ell t,\ell}$	Coefficients of mutual influence relative to material axes
θ	Temperature
κ	Thermal diffusivity
$\nu_{\ell t_k}, \nu_{t\ell_k}$	Major and minor Poisson's ratios of k'th lamina, respectively, relative to the material axes
$\sigma_{\ell_k}, \sigma_{t_k}$	Median-surface extensional stresses of k'th lamina relative to the material axes
$\sigma_{x_k}, \sigma_{y_k}$	Median-surface extensional stresses of the k'th lamina relative to the plate axes
$\sigma_x^{(0)}, \sigma_x^{(1)} \ldots$ $\sigma_y^{(0)}, \sigma_y^{(1)} \ldots$	Laminate averaged extensional stresses relative to plate axes
$\tau_{\ell t_k}$	Median-surface shearing stress of k'th lamina relative to material axes
τ_{xy_k}	Median-surface shearing stress of k'th lamina relative to plate axes

$\tau_{xy}^{(0)}, \tau_{xy}^{(1)} \ldots$	Laminate averaged inplane shear stress relative to plate axes
$\vec{\tau}_{yz}, \vec{\tau}_{zx}$	Interlaminar shear stress vectors
$\left.\begin{array}{l}\tau_{yz}^{(0)}, \tau_{yz}^{(1)} \ldots \\ \tau_{zx}^{(0)}, \tau_{zx}^{(1)} \ldots\end{array}\right\}$	Laminate averaged transverse shearing stresses relative to plate axes

PROJECT SUMMARY

CONTRACT NUMBER: F33615-86-C-3218

CONTRACTOR: Anamet Laboratories, Inc.
3400 Investment Blvd.
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PRINCIPAL INVESTIGATOR: Rocky Richard Arnold, Ph.D.

TITLE OF THE PROJECT: Rapid Thermal Loading of Delaminated Composite Structures

TECHNICAL DESCRIPTION

The research program proposed herein was directed toward developing a theoretical approach to the problem of rapid thermal loading of delaminated composite structure. The structures examined include flat and shallow-curved plates and cylindrical shells. Using Hamilton's principle and the Reissner variational theorem, a new dynamic thermoelastic variational principle was developed. Application of this principle to both plates and shells provided the equations of dynamic equilibrium and the associated natural and geometric boundary conditions. Important physically observable characteristics of composites, not usually found in classical approaches, were also included in the theory. These characteristics include transverse shear, material nonlinearity and delamination mechanisms.

In this Phase I effort, the thermoelastic model was defined and the governing equations derived for both contiguous and delaminated plate and shell structures. Solution of the governing equations and numerical examples constitutes the major emphasis of the Phase II work.

COMMERCIAL APPLICATION

The most immediate benefit from the proposed effort will be the ability to design/analyze delaminated composite structures with improved resistance to rapid thermal loading. This has direct application to various military systems, especially aircraft that may be subject to hostile laser threats. Other potential applications include future hypersonic vehicles and civilian transport aircraft, wherein lower rates of thermal loading are of concern.

1.0 INTRODUCTION

This is the Phase I final report for work accomplished under U.S. Air Force Contract No. F33615-86-C-3218, as a part of the Federally-sponsored Small Business Innovation Research (SBIR) program. The period of technical performance was from July 31, 1986 to January 31, 1987. The research and development effort documented herein was accomplished for AFWAL/GLXPF.

The primary objective of the Phase I research was to derive a set of governing equations for both contiguous and delaminated composite plates and shells subject to intense rapid thermal heating. These equations are amenable to programming on a digital computer, which is the primary emphasis of any potential Phase II work. The objective of this research has been successfully accomplished and the assumptions and derivations used to complete this project are documented within this final report.

2.0 THEORY

The basic structural configurations under consideration herein are the flat plate, shallow-curved plate, and complete circular shell. Schematic diagrams of these basic aircraft structures and the coordinate systems used in the ensuing derivations are included in Figure 1. For all structural configurations, the laminated composite is examined in two distinctly different states; that is, either the composite is undamaged (contiguous state), or the composite is delaminated presumably through the action of a rapid thermal pulse. In both cases, the primary theoretical derivations are identical--differences between either the contiguous or delaminated composite configurations are manifested by the appropriate choice of stress and displacement functions which will be explained in more detail in a latter part of this report (Section 3.1).

In this section, the basic theoretical derivations, valid for both contiguous and delaminated composite configurations, are developed and limiting assumptions stated.

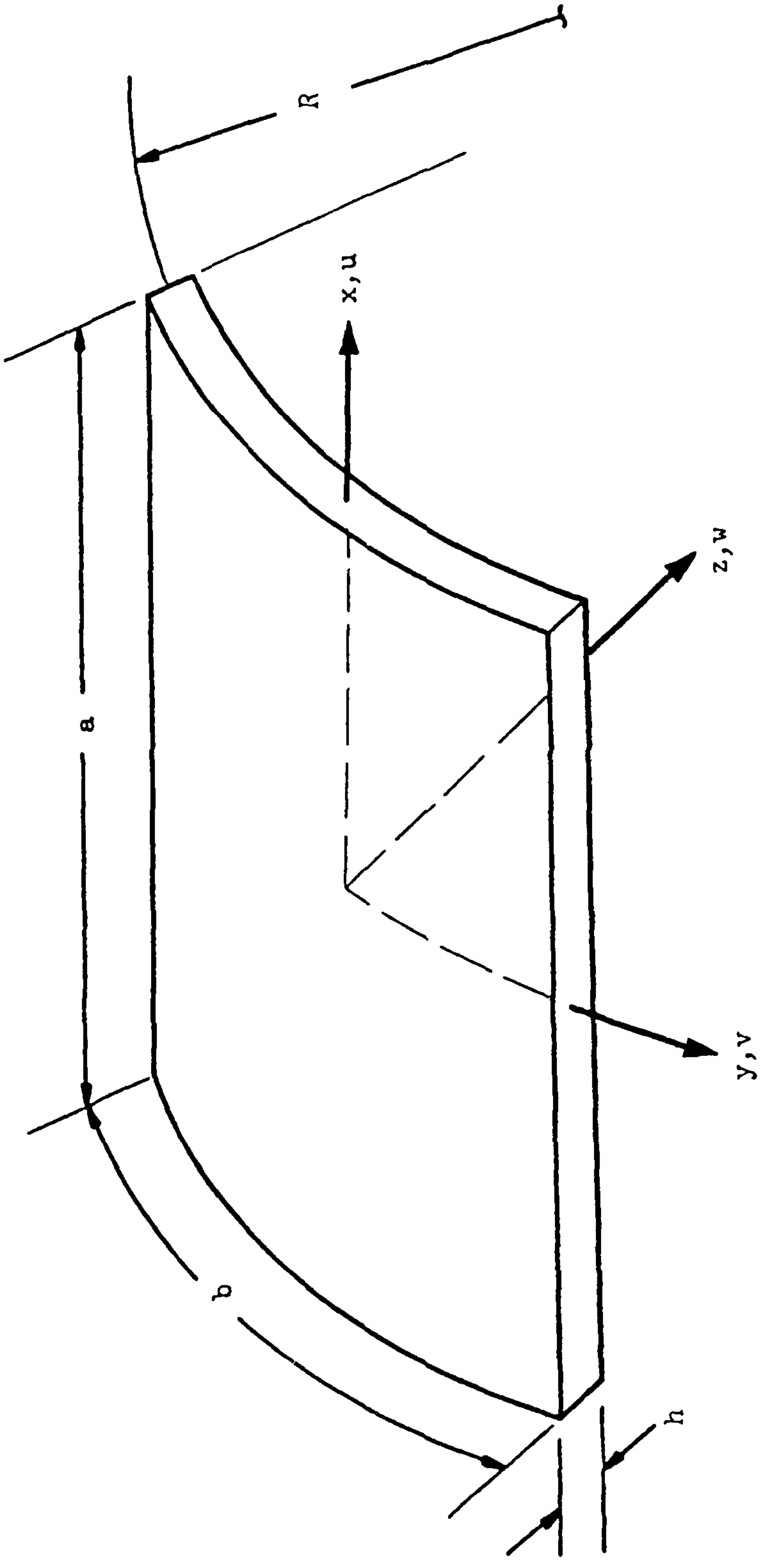

Figure 1 Curved Element Geometry and Coordinate System

2.1 Statement of Problem and Principal Assumptions

The plate and shell elements consist of N homogeneous and materially nonlinear anisotropic laminae with each lamina having thickness t_{fm_k}. Each element of a lamina midsurface can undergo three translations, two inplane displacements, u_k and $v_k (k=1,2,...N)$, and a lateral displacement w which is common to all laminae. The inplane stresses σ_{x_k}, σ_{y_k} and τ_{xy_k} are assumed constant through the thickness of each lamina.

The bending stiffness of an individual lamina is assumed negligible compared to that of the laminate; that is, laminae are assumed to behave as membranes. Consequently, transverse shear effects are accommodated by permitting relative movement between the median planes of adjacent laminae. The matrix material between adjacent laminae midsurfaces is assumed to carry all of the transverse shear in the y-z and z-x planes. The normal stress (σ_z) is taken to be negligible in comparison with the other interlaminar stresses (τ_{yz}, τ_{zx}) for the composites plates considered herein; namely, those that buckle in the linear-elastic range and become elasto-plastic in the postbuckling range.

The assumptions cited above are known to be valid for the contiguous composite [1]; however, for a delaminated composite the normal stress (σ_z) that would exist at the interface between adjacent laminae that contain the delamination is not necessarily negligible. During the Phase II work, a closer examination of this assumption will be accomplished and, if required, suitable modifications made to the approach proposed herein.

2.2 Strain-Displacement Relations

The ultimate success of any theoretical model is the appropriate definition of initial assumptions--these initial assumptions determine the accuracy and utility of the finally-developed analytical method. For this research, the membrane strain-displacement relations of Sanders [2] are used. For a circular cylindrical shell, these relations are:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{1}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \underline{\frac{v}{R}}\right)^2 \tag{2}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \underline{\frac{v}{R}\frac{\partial w}{\partial x}} \tag{3}$$

For shallow-curved shells, the underlined terms are ignored and in the case of flat plates, the radius becomes infinite ($R \to \infty$).

Sanders' complete strain-displacement expressions are valid when (1) the Kirchoff-Love hypothesis holds, (2) middle surface strains and rotations out of the middle surface are small in comparison to unity (note: the displacement components are not necessarily small), and (3) rotations about the normals to the middle surface are small in comparison to rotations out of the middle surface.

To allow for the inclusion of transverse strain in the variational formulation, Equations 1-3 are used to define the membrane strains in each individual lamina, and the lamina summations process provides the proper strain-displacement relations in bending (much in the same way classical lamination theory is developed). This approach has been used previously in References 1,3-5 wherein a unique model for inclusion of transverse shear in laminated composites has been shown to be very accurate in the prediction of initial buckling and plate bending problems.

When the inplane and bending strain-displacement relations are decoupled in this manner, the Kirchoff-Love hypothesis is effectively removed as an initial assumption and the transverse shear strain-displacement relations become

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{u_{k+1} - u_k}{t_{m_k}} \tag{4}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{v_{k+1} - v_k}{t_{m_k}} \tag{5}$$

2.3 Constitutive Relations

The nonlinear elastic stress-strain relations used (in material coordinates) are based on those of Reference 1 and are shown below.

$$\varepsilon_{\ell_k} = \left\{\frac{\sigma_\ell}{E_\ell} - \frac{\nu_{\ell t}}{E_\ell}\sigma_t + \frac{\eta_{\ell,\ell t}}{G_{\ell t}}\tau_{\ell t} + K_\ell\left(\frac{\sigma_\ell}{E_\ell}\right)^{n_\ell}\right\}_k \tag{6}$$

$$\varepsilon_{t_k} = \left\{\frac{\sigma_t}{E_t} - \frac{\nu_{t\ell}}{E_t}\sigma_\ell + \frac{\eta_{t,\ell t}}{G_{\ell t}}\tau_{\ell t} + K_t\left(\frac{\sigma_t}{E_t}\right)^{n_t}\right\}_k \tag{7}$$

$$\gamma_{\ell t_k} = \left\{\frac{\tau_{\ell t}}{G_{\ell t}} + \frac{\eta_{\ell t,\ell}}{E_\ell}\sigma_\ell + \frac{\eta_{\ell t,t}}{E_t}\sigma_t + K_{\ell t}\left(\frac{\tau_{\ell t}}{G_{\ell t}}\right)^{n_{\ell t}}\right\}_k \tag{8}$$

$$\gamma_{yz_k} = \left\{\frac{\tau_{yz}}{Gm} + K_m\left(\frac{\tau_{yz}}{G_m}\right)^{n_m}\right\}_k \tag{9}$$

$$\gamma_{zx_k} = \{\frac{\tau_{zx}}{G_m} + K_m(\frac{\tau_{zx}}{G_m})^{n_m}\}_k \tag{10}$$

Equations 9 and 10 reflect the transverse shear present in the model. Note, in each equation a Ramberg-Osgood [6] type formulation is used to describe the inelastic behavior of the composite material. These relations or very similar ones have been used in the solution of several different types of problems [7,8], and in each case, the resulting numerical model has been shown to be quite accurate and efficient.

2.4 Modified Hamilton/Reissner Principle

The basis of the analytical procedure developed during this research is the extension of Hamilton's principle from rigid to deformable bodies. Hamilton's principle is stated as,

$$\delta = \int_{t_1}^{t_2} (T - U - V)dt = 0 \tag{11}$$

where t_1 and t_2 are arbitrary time values, T is the kinetic energy and U+V is the total potential energy of the system. The kinetic energy is the sum of kinetic energies associated with each of the orthogonal velocities (axial, tangential, and radial); that is,

$$T = \frac{1}{2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \rho[(\frac{\partial u}{\partial t})^2 + (\frac{\partial v}{\partial t})^2 + (\frac{\partial w}{\partial t})^2]dzdydx \tag{12}$$

where ρ is the mass density of the material. Now, if Hamilton's principle is modified by replacing the strain energy U with the Reissner functional U" [9], the variational equation becomes

$$\delta = \int_{t_1}^{t_2} (T - U'')dt = 0 \tag{13}$$

where the Reissner functional is

$$U'' = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} F'' \, dzdydx \tag{14}$$

and F" is the Reissner functional

$$F'' = \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} - F'_{fm} \tag{15}$$

The quantity F' is the stress-energy-density function which for a laminae is given by

$$F'_{fm_k} = F'_{fm_{e_k}} + F'_{fm_{p_k}} \tag{16}$$

where

$$F'_{fm_{e_k}} = \frac{1}{2}\left\{\frac{\sigma_\ell^2}{E_\ell} + \frac{\sigma_t^2}{E_t} - \left(\frac{\nu_{\ell t}}{E_\ell} + \frac{\nu_{t\ell}}{E_t}\right)\sigma_\ell\sigma_t + \left(\frac{\eta_{\ell,\ell t}}{G_{\ell t}} + \frac{\eta_{\ell t,\ell}}{E_\ell}\right)\sigma_\ell\tau_{\ell t}\right.$$

$$\left. + \left(\frac{\eta_{t,\ell t}}{G_{\ell t}} + \frac{\eta_{\ell t,t}}{E_t}\right)\sigma_t\tau_{\ell t} + \frac{\tau_{\ell t}^2}{G_{\ell t}}\right\}_k \tag{17}$$

and

$$F'_{fm_{p_k}} = \left\{\frac{K_\ell}{(n_\ell+1)E_\ell} \varepsilon_{eff}^{(n_\ell-1)} \left(\sigma_\ell^2 - \nu_{\ell t}\sigma_\ell\sigma_t + \eta_{\ell t,\ell}\sigma_\ell\tau_{\ell t}\right)\right.$$

$$+ \frac{K_t}{(n_t+1)E_t} \varepsilon_{eff}^{(n_t-1)} (\sigma_t^2 - \nu_{t\ell}\sigma_\ell\sigma_t + \eta_{\ell t,t}\sigma_t\tau_{\ell t})$$

$$+ \frac{1}{2}\left[\frac{3}{4} \cdot \frac{1}{(1+\nu_{\ell t})} \cdot \frac{1}{(1+\nu_{t\ell})}\right]^{(1-n_{\ell t})/2}$$

$$\cdot \frac{K_{\ell t}}{(n_{\ell t}+1)G_{\ell t}} \varepsilon_{eff}^{(n_{\ell t}-1)} (\tau_{\ell t}^2$$

$$+ \eta_{t,\ell t}\sigma_t\tau_{\ell t} + \eta_{\ell,\ell t}\sigma_\ell\tau_{\ell t})\}_k$$

The effective strain is defined by

$$\varepsilon_{eff_k} = \{(\frac{\sigma_\ell}{E_\ell})^2 + (\frac{\sigma_t}{E_t})^2 - \frac{(\nu_{\ell t} + \nu_{t\ell})}{E_\ell E_t} \sigma_\ell\sigma_t$$

$$+ 3 \cdot \frac{1}{2(1+\nu_{\ell t})} \cdot \frac{1}{2(1+\nu_{t\ell})} [(\frac{\tau_{\ell t}}{G_{\ell t}})^2$$

$$+ (\frac{\eta_{\ell,\ell t}+\eta_{\ell t,\ell}}{E_\ell G_{\ell t}})\sigma_\ell\tau_{\ell t} + (\frac{\eta_{t,\ell t}+\eta_{\ell t,t}}{E_t G_{\ell t}})\sigma_\ell\tau_{\ell t}]\}_k^{1/2} \qquad (18)$$

For transverse shear

$$F'_{m_k} = F'_{m_{e_k}} + F'_{m_{p_k}} \qquad (19)$$

where

$$F'_{m_{e_k}} = \left\{\frac{\tau_{yz}^2}{2G_m} + \frac{\tau_{zx}^2}{2G_m}\right\}_k \tag{20}$$

and

$$F'_{m_{p_k}} = \left\{\frac{K_m}{(n_m+1)} \cdot \left(\frac{1}{G_m}\right)^{n_m} \cdot \left(\tau_{yz}^2 + \tau_{zx}^2\right)^{(n_m+1)/2}\right\}_k \tag{21}$$

where Equations 6, 7, and 8 apply to inplane behavior and Equations 9 and 10 are applicable to transverse shear in the laminate. It should be noted that the derivative of the quantity F' with respect to any stress component provides the required strain component.

By combining the strain-displacement relations and constitutive relations of Sections 2.2 and 2.3 with the modified Hamilton's principle, a complete new variational problem is defined. The resulting modified Hamilton/Reissner functional is presented in Appendix A in vector-matrix form [10]. It is stated, but not proven herein, that the first variation of the modified Hamilton/Reissner functional provides both the governing equations of dynamic equilibium along with the associated stress-displacement boundary conditions and the constitutive relationships.

2.5 Temperature-Distribution History

For each kind of structure examined in this study, the thermal forcing function is a suddenly applied uniform heat flux acting over the surface z = -h/2 (see Figure 1) while surface

$z = h/2$ is presumed to be perfectly insulated. These conditions are exactly those analyzed by Boley and Barber [11], Kraus [12], and Lu and Sun [13]. They are selected here for convenient comparison of present results with those of previous analyses.

The temperature distribution history, given by solution of the one-dimensional Fourier heat conduction equation, is

$$\theta(z,t) = \frac{qh}{k}\left[\frac{\kappa}{h^2} t + \frac{1}{2}\left(\frac{1}{2} - \frac{z}{h}\right)^2 - \frac{1}{6}\right.$$

$$\left. - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-\kappa(n\pi/h)^2 t} \cos n\pi\left(\frac{1}{2} - \frac{z}{h}\right)\right] \qquad (22)$$

where q is the heat flux, h is the thickness of the shell, k is the thermal conductivity, and κ is the thermal diffusivity. In Figure 2, the temperature-distribution through the thickness corresponding to $q = 10$ is plotted at intervals of 1/100 of a thermal period for a duration of one thermal period.

3.0 METHOD OF SOLUTION

Extremization of the Reissner functional (Appendix B) produces the governing equilibrium and constitutive equations and the consistent natural and/or geometric boundary conditions. Once these governing equations are obtained, it is necessary to effect a solution; that is, determine the stresses and displacements throughout the volumes. The exact solution of the nonlinear system of governing equations is, in general, prohibitive. Thus, approximation methods must be used. This being the case, the extremization of the modified Hamilton/Reissner functional may be used in more direct fashion to effect an approximate solution. Since the vanishing of the first variation of the modified Hamilton/Reissner functonal leads to the governing system of

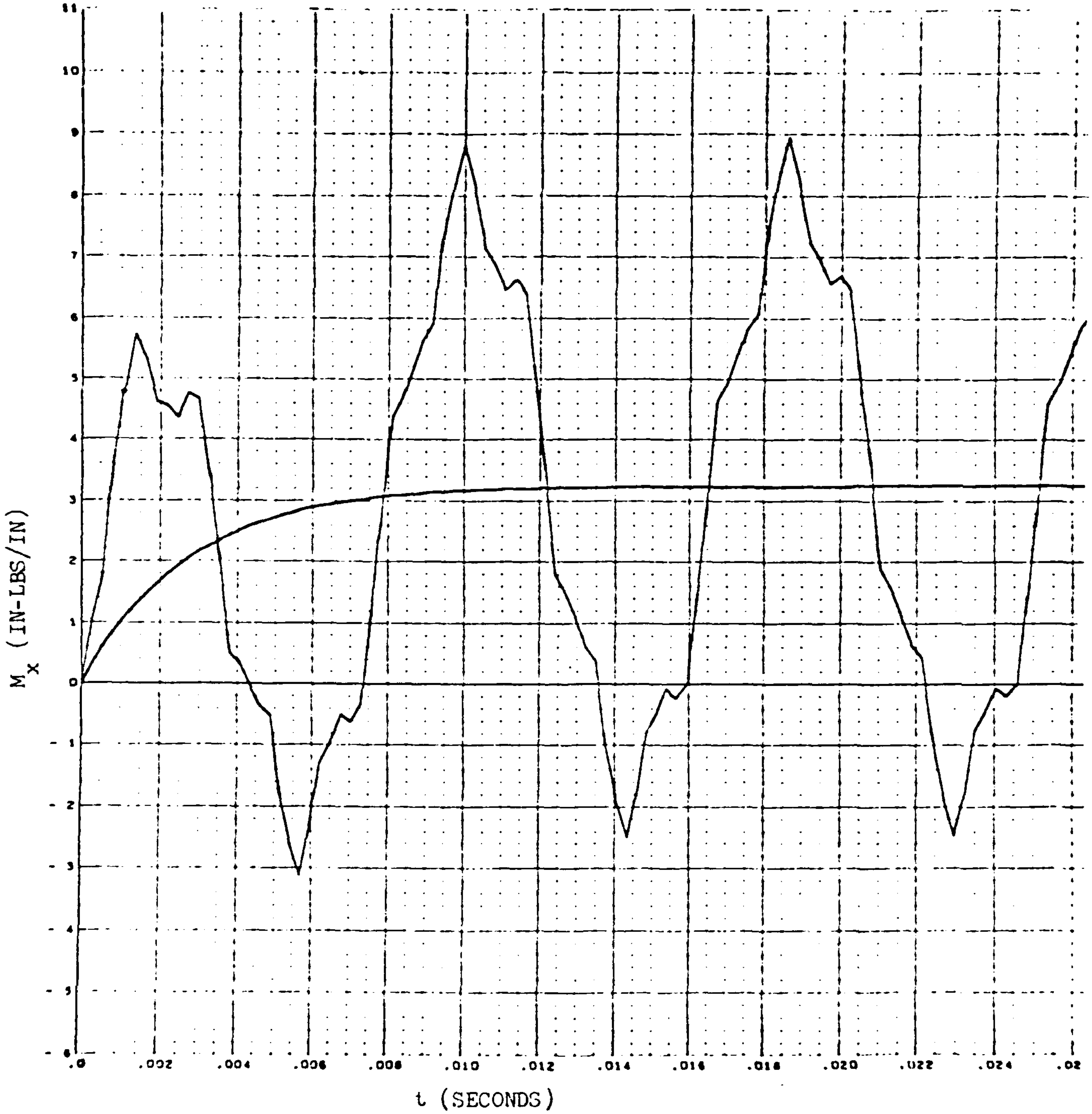

Figure 2 Boley and Barber's Histories of Thermoelastic and Total Moment at the Center of the Plate

equations and boundary conditions, then it is possible to assume a spatial distribution and time dependence of the variational unknowns satisfying desired boundary conditions, integrate out the spatial dependence, and obtain a system of differential equations for the time-dependent variational parameters corresponding the approximate solution of the governing equations. The resulting system of nonlinear time-dependent differential equations may be solved by any number of numerical procedures (for example, Newton-Raphson for converging to a solution at any particular time step and Runge-Kutta [14] for solution in the time domain). The method of solution is outlined in a subsequent section.

Application of the Newton-Raphson iterative technique results in the set of equations contained in Appendix C. The principal unknowns are the stresses ($\vec{h}_1, \vec{h}_2, \vec{h}_3, \vec{a}, \vec{b}$) and displacements ($\vec{e}, \vec{f}$, and $\vec{g}$), all of which are time dependent. Once the spatial variation of the variational unknowns is specified (Section 3.1), the integrations of Appendices B and C can be performed. The resulting set of equations are symbollically written as

$$\tilde{A}^* \, \Delta\vec{X} = \tilde{A} \, \vec{X}^* + \vec{F}_{ext} + \tilde{M}^* \, \ddot{\vec{X}}^* \tag{23}$$

where the individual terms are expanded in Appendix D.

3.1 Displacement and Stress Functions

In the more classical procedures, either Minimum Potential Energy or Complementary Energy, the variational unknowns are either the displacements or the stresses, respectively. The final equations involve a relatively small number of unknowns because either the constitutive relations or the equilibrium equations have been used to reduce the overall number of governing equations to be solved. As a consequence, the final equations, when viewed in a vector-matrix form, wherein the vector of unknown quantities are either displacement or stress amplitudes, have coupling matrices which are virtually fully-populated with terms; i.e., the coupling is strong. However, when the Reissner variational approach is used, the vector of unknown quantities are both displacements and stresses representing both equilibrium and stress-displacement compatibility. The coupling matrix between the variational unknowns is observed to be very sparse; i.e., loosely coupled.

This loose coupling is a problem because an inappropriate choice of displacement and stress functions can lead to a situation wherein the computed integral that couples the physical terms can be zero--this would imply that the computational procedure would not recognize the physical coupling that is present in the real-life structure. As a consequence of this mathematical consideration, it is necessary to select a complete set of functions (displacement and stress) that admit coupling between physical quantities that should be coupled.

It should also be noted that it is possible to select functions that provide for coupling when it should not occur. This can also lead to fictitious results. In any event, the first and foremost requirement to be placed on the selected displacement and stress functions is that the proper kind of coupling be maintained between the physical entities of displacement and stress.

The next requirement to be satisfied by the distribution functions is that they represent the physical behavior of the structure under analysis. In practice this is usually not too difficult in the sense that it is usually easy to select either trigonometric or polynomic functions that match approximately the expected displacements and stresses of the structure to be analyzed. In this sense, for complicated problems, wherein the exact form of the distributions is not known (as with the present problem), it is necessary to select several "generic" forms and experiment with the combined distributions to determine which ones provide viable solutions without disturbing the need for proper coupling as discussed previously. At this point, the selection and verification of displacement and stress functions used to solve the problem of the buckling, postbuckling, and crippling of laminated composite shallow shells under combined axial compression and shear [5] took approximately seven months of intense effort in spite of the fact that the initial selection of trial functions was achieved in approximately two weeks. The issue of loose coupling was a major factor in the time required.

Although not mentioned yet, but just as important, there are additional requirements that are placed on both displacement and stress functions. For instance, displacement functions must satisfy the boundary conditions of the problem under consideration. Stress functions must satisfy equilibrium across cross-sections as a minimum (based upon the experience of the principal investigator), although this requirement can be waived under the proper circumstances.

It should be remembered that in using the Reissner variational principle, the constitutive relations are not satisfied concurrently at every point in the structure. At first thought, this rather pointed observation is disturbing because most knowledgeable enigneers familiar with mechanics would state that the constitutive relation must be satisfied at every point for the solution to be considered correct. In actual practice this presents no particular problem if some ordering of priorities is kept in mind.

First, the objective in using the Reissner variational approach is to obtain an accurate knowledge of the state of displacement in the structure at the minimal computational cost. This is usually achieved by careful selection of a complete but modest number of displacement functions which satisfy the boundary conditions. Stress functions are then chosen to satisfy equilibrium across some cross-section or area, but not at every point. These same stress functions must couple correctly with the displacement functions in such a way that the finally computed displacement amplitudes and distributions are accurate. In actual practice it takes a combination of knowledge, intuition, and perseverance to achieve this end--but it is possible, as documented in earlier similar efforts [3,4,5].

In the end, when the numerical results are produced, the stress amplitudes are essentially discarded in the sense that the numerical values of stress may or may not be accurate--it is not important because the goal of the approach is to obtain accurate information about displacement. From displacements the strains may be readily calculated because the strain-displacement relations are in general accurate and a single derivative does not lead to gross inaccuracies. Stresses, of an accurate nature, can be obtained via the use of the original complete set of nonlinear constitutive relations.

The discussions of the preceding paragraphs give an idea of the difficulties associated with the selection of displacement and stress functions for use in the Reissner variational principle. In general, these comments apply to any particular problem; however, for this particular research concerned with the effects of delaminations, there are some additional considerations that must be accounted for. In addition to the unique application of the Reissner and Hamilton principles, the use of the direct variational approach to solve a problem which inherently couples local and global displacement/stress fields is also noted as a significant challenge.

It is anticipated that interactions between local and global displacement/stress fields can occur deleteriously, as discussed previously. Thus, selection of global and local functions is potentially a difficult and time consuming process even though the starting functions seem intuitively obvious; i.e., the local stress fields near a crack-tip representing a delamination are generally known.

3.1.1 Contiguous Composite

Based on the earlier work of Stroud and Mayers [15], the displacements are taken as

$$u = u_1 \zeta \tag{24}$$

$$v = v_1 \eta + v_2 (1 - \zeta^2)(\eta - \frac{1}{3}\eta^3) \tag{25}$$

$$w = w_{22}(1-\zeta^2)(1-\eta^2) + w_{24}(1-\zeta^2)(\eta^2-\eta^4) + w_{42}(\zeta^2-\zeta^4)(1-\eta^2) \tag{26}$$

where ζ and η are dimensionless goemetric coordinate parameters defined as

$$\zeta = 2x/a \tag{27}$$

and

$$\eta = 2y/b \tag{28}$$

For flat plates, the second term for the expression for ν is omitted.

It is apparent that for the linear thermoelastic response of a plate with unrestrained edges, the s tions assumed for inplane displacements are exact. The transverse-displacement expression is a fourth-order polynomial with all except three of the coefficients eliminated by enforcing geometric boundary conditions and symmetry requirements (that is, vanishing displacement at the supports and symmetry about the coordinate axes).

The distributions of bending and twisting moments assumed for the thermoelastic solution are

$$M_{xB} = - M_T + \frac{\overline{C}}{\overline{H}} N_T + \left[(1-\nu)(M_T - \frac{\overline{C}}{\overline{H}} N_T) - M_{x22}\right](1-\zeta^{10})\eta^2$$

$$+ M_{x22}[1 - \zeta^2(1-\eta^{10})] + M_{x24}(1-\zeta^2)(\eta^2-\eta^4) + M_{x42}(\zeta^2-\zeta^4)(1-\eta^2)$$

(29)

$$M_{yB} = -M_T + \frac{\overline{C}}{\overline{H}} N_T + [(1-\nu)(M_T - \frac{\overline{C}}{\overline{H}} N_T) - M_{y22}]\zeta^2(1-\eta^{10})$$

$$+ M_{y22}[1-(1-\zeta^{10})\eta^2] + M_{y24}(\zeta^2-\zeta^4)(1-\eta^2) + M_{y42}(1-\zeta^2)(\eta^2-\eta^4)$$

(30)

$$M_{xyB} = M_{xy11} \sin\frac{\pi}{2}\zeta \sin\frac{\pi}{2}\eta + M_{xy13} \sin\frac{\pi}{2}\zeta \sin\frac{3\pi}{2}\eta$$

$$+ M_{xy31} \sin\frac{3\pi}{2}\zeta \sin\frac{\pi}{2}\eta \qquad (31)$$

3.1.2 Delaminated Composite

For a delaminated composite specimen the out-of-plane displacement w is, in general, a function of the through thickness coordinate, z, as well as the other spatial coordinates and time; that is, $w = w(z,x,y,t)$. Equivalently, the out-of-plane displacement is considered to be a function of the lamina identification, k; thus, each lamina can have a different functional form for the out-of-plane displacement. For instance, in the local region around the delamination, a potential choice for displacement function is

$$w_{local} = w_d\left(1 + \cos \frac{\pi(\bar{x}^2 + \bar{y}^2)}{a^2}\right)$$

where w_d is the amplitude of the delamination, $\bar{x}$ and $\bar{y}$ are spatial coordinates located at the center of the assumed circular delamination of radius a, and

$$w = w_{global} + w_{local}$$

where w_{global} was, for a contiguous composite, the same as w.

Corresponding to this out-of-plane displacement are local stress fields analogous to those near a crack tip (see Sih [16], for instance).

3.2 Iterative Solution Technique in the Time Domain

When, at any one time step subsequent to updating material properties and temperatures, the governing equations are solved, then $\Delta\vec{X} \to 0$ and the resulting differential equation is

$$\tilde{M}\ddot{\vec{X}}^* + \tilde{A}\vec{X} + F_{ext} = 0 \tag{32}$$

Defining

$$\vec{y} = \dot{\vec{X}}^* \tag{33}$$

then

$$\tilde{M}\dot{\vec{y}} + \tilde{A}\vec{X} + F_{ext} = 0 \tag{34}$$

Equations (33) and (34) can be written as a set of first order differential equations:

$$\begin{bmatrix} \tilde{M} & 0 \\ 0 & \tilde{I} \end{bmatrix} \begin{Bmatrix} \dot{\vec{y}} \\ \dot{\vec{x}} \end{Bmatrix} + \begin{bmatrix} \tilde{A} & 0 \\ 0 & -\tilde{I} \end{bmatrix} \begin{Bmatrix} \vec{x} \\ \vec{y} \end{Bmatrix} + \begin{bmatrix} \vec{F}_{ext} \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} \tag{35}$$

The above equation can be solved with the use of the Runge-Kutta [14] approach.

4.0 RESULTS AND CONCLUSIONS

The objective of this Phase I research was to derive the equations governing the response of a delaminated composite to a rapid thermal pulse. The approach taken was one of developing a variational formulation based on a unique coupling of Hamilton's and Reissner's variational theorems. The resulting equations have been cast into a form that is suitable for programming on a digital computer. The solution of these equations provides a description of the state of displacement and stress within the delaminated composite.

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APPENDIX A

MODIFIED-HAMILTON/REISSNER FUNCTIONAL IN VECTOR-MATRIX NOTATION

Hamilton's principle is stated as

$$\delta \int_{t_1}^{t_2} (T - U - V)dt = \delta \int_{t_1}^{t_2} (T - U")dt = 0 \qquad \text{(A-1)}$$

where the Reissner functional in vector-matrix notation [10] is,

$$U" = \int_x \int_y \{\vec{h}_1^T \tilde{\lambda}_1^T \tilde{P}_1^T \tilde{t}_{fm}\{\vec{I}\ \frac{1}{2}\ \vec{g}^T \frac{\partial\vec{\gamma}}{\partial x} \frac{\partial\vec{\gamma}}{\partial x} \vec{g} + \delta\vec{I} + \tilde{Q}_x \frac{\partial\vec{\alpha}}{\partial x} \vec{e}\}$$

$$+ \vec{h}_2^T \tilde{\lambda}_2^T \tilde{P}_2^T \tilde{t}_{fm}\{\vec{I}\ \frac{1}{2}\ \vec{g}^T \frac{\partial\vec{\gamma}}{\partial y} \frac{\partial\vec{\gamma}^T}{\partial y} \vec{g} + \tilde{Q}_y \frac{\partial\tilde{\beta}}{\partial y} \vec{f}$$

$$- \vec{h}_2^T \tilde{\lambda}_2^T \tilde{P}_2^T \tilde{g}^T \tilde{t}_{fm}\ \vec{\gamma}\ \frac{1}{R} + \vec{f}^T\ \tilde{\beta}^T\ \tilde{Q}_y^T\ \tilde{t}_{fm}\ \tilde{P}_2\ \tilde{\lambda}_2\ \vec{h}_2 \frac{\partial\vec{\gamma}^T}{\partial y} \vec{g}\ \frac{1}{R}$$

$$+ \vec{f}^T\ \tilde{\beta}^T\ \tilde{Q}_y^T\ \tilde{P}_2\ \tilde{\lambda}_2\ \vec{h}_2\ \vec{f}^T\ \tilde{\beta}^T\ \tilde{Q}_y^T \frac{1}{R^2}\}$$

$$+ h_3^T \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm}\{\vec{I}\ \vec{g}^T(\frac{\partial\vec{\gamma}}{\partial x} \frac{\partial\vec{\gamma}^T}{\partial y} \vec{g}) + \tilde{Q}_x \frac{\partial\tilde{\alpha}}{\partial y} \vec{e} + \tilde{Q}_y \frac{\partial\tilde{\beta}}{\partial x} \vec{f}$$

$$+ \tilde{g}^T \frac{\partial\vec{\gamma}}{\partial x} \vec{h}_3^T \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm}\ \tilde{Q}_y\ \tilde{\beta}\ \vec{f}\}$$

$$+ \vec{a}^T\ \tilde{\varepsilon}^T\ \tilde{R}_x^T\{\vec{I}\ \frac{\partial\vec{\gamma}^T}{\partial y} \vec{g} + \tilde{B}\ \tilde{Q}_x\ \tilde{\alpha}\ \vec{e}\}$$

$$+ \vec{b}^T\ \tilde{\eta}^T\ \tilde{R}_y^T\{\vec{I}\ \frac{\partial\vec{\gamma}^T}{\partial x} \vec{g} + \tilde{B}\ \tilde{Q}_y\ \tilde{\beta}\ \vec{f}\}$$

$$- \sum_{L=1}^{3} \sum_{M=1}^{3} \vec{h}_m^T\ \tilde{\lambda}_M^T\ \tilde{P}_M^T\ \tilde{C}_{LM}\ \tilde{P}_L\ \tilde{\lambda}_L\ \vec{h}_L - \alpha_x \Delta T \tilde{P}_1 \tilde{\gamma}_1 \vec{h}_1 - \alpha_y \Delta T \tilde{P}_2 \tilde{\lambda}_2 \vec{h}_2$$

$$- \vec{a}^T \tilde{\varepsilon}^T \tilde{R}_x^T \bar{G}_m \tilde{R}_x \tilde{\varepsilon} \vec{a}$$

$$- \vec{b}^T \tilde{\eta}^T \tilde{R}_y^T \bar{G}_m \tilde{R}_y \tilde{\eta} \vec{b}$$

$$- \sum_{L=1}^{3} \sum_{M=1}^{3} \vec{h}_M^T \tilde{\lambda}_M^T \tilde{P}_M^T \sum_{k=1}^{N} \sum_{i=1}^{3} \sum_{j=1}^{3} \vec{T}_{iL_k} (2\delta_{ij-1})$$

$$\cdot \left[\sum_{\ell=1}^{3} \sum_{m=1}^{3} \vec{h}_\ell^T \tilde{\lambda}_\ell^T \tilde{P}_\ell^T \tilde{D}_{\ell m i j_k} \tilde{P}_m \tilde{\lambda}_m \vec{h}_m \right]^{p_{1_k}} \vec{T}_{jm_k} \tilde{P}_L \tilde{\lambda}_L \vec{h}_L$$

$$- \sum_{k=1}^{(N-1)} (\vec{a}^T \tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{R}_x \tilde{\varepsilon} \tilde{a} + \vec{b}^T \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{R}_y \tilde{\eta} \vec{b})^{p_4} \} dxdy \tag{A-2}$$

and similarly, the kinetic energy is

$$T = \frac{1}{2} \int_x \int_y \left\{ \frac{\partial \vec{e}^T}{\partial t} \tilde{\alpha}^T \tilde{Q}_x^T \tilde{t}_{fm} \tilde{Q}_x \tilde{\alpha} \frac{\partial \vec{e}}{\partial t} + \frac{\partial \vec{f}^T}{\partial t} \tilde{\beta}^T \tilde{Q}_y^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta} \frac{\partial \vec{f}}{\partial t} \right.$$

$$\left. + \frac{\partial \vec{g}^T}{\partial t} \vec{\gamma}^T \vec{I}_N \tilde{I}_N^T \tilde{t}_{fm} \vec{\gamma} \frac{\partial \vec{g}}{\partial t} \right\} dxdy \tag{A-3}$$

APPENDIX B

FIRST VARIATION OF THE MODIFIED-HAMILTON/REISSNER FUNCTIONAL

The governing equations which result from taking the first variation of the modified Hamilton/Reissner functional (Eq. (A-1)) are as follows:

$$\frac{1}{2} \int_x \int_y \tilde{\lambda}_1^T \tilde{P}_1^T \vec{I}_N \vec{g} \frac{\partial \vec{\gamma}}{\partial x} \frac{\partial \vec{\gamma}^T}{\partial x} \tilde{t}_{fm} \vec{g} \, dxdy$$

$$+ \tilde{\phi}_1 \vec{e} + \sum_{L=1}^{3} \tilde{\phi}_{1L} \vec{h}_L + \vec{G}_1 = \vec{B} + \vec{B}_1 \qquad \text{(B-1)}$$

$$\frac{1}{2} \int_x \int_y \tilde{\lambda}_2^T \tilde{P}_2^T \vec{I}_N \vec{g} \frac{\partial \vec{\gamma}}{\partial y} \frac{\partial \vec{\gamma}^T}{\partial y} \tilde{t}_{fm} \vec{g} \, dxdy$$

$$+ \tilde{\phi}_2 \vec{f} + \sum_{L=1}^{3} \tilde{\phi}_{2L} \vec{h}_L + \vec{G}_2 + \tilde{\lambda}_2^T \tilde{P}_2^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta} \vec{f} \frac{\partial \vec{\gamma}^T}{\partial y} \vec{g} \frac{1}{R} \qquad \text{(B-2)}$$

$$+ \tilde{\lambda}_2^T \tilde{P}_2^T \vec{I}_N \vec{f}^T \tilde{\beta}^T \tilde{Q}_y^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta} \vec{f} \frac{1}{R^2} = \vec{B}_2$$

$$\frac{1}{2} \int_x \int_y \tilde{\lambda}_3^T \tilde{P}_3^T \vec{I}_N \vec{g} \frac{\partial \vec{\gamma}}{\partial x} \frac{\partial \vec{\gamma}^T}{\partial y} \tilde{t}_{fm} \vec{g} \, dxdy$$

$$+ \vec{\phi}_3 \vec{e} + \tilde{\phi}_4 \vec{f} + \sum_{L=1}^{3} \tilde{\phi}_{3L} \vec{h}_L + \vec{G}_3 + \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta} \vec{f} \frac{\partial \tilde{\gamma}^T}{\partial x} \vec{g} \frac{1}{R} = \vec{0}$$

$$\text{(B-3)}$$

$$\int_x \int_y \{\frac{\partial\vec{\gamma}}{\partial x}\frac{\partial\vec{\gamma}^T}{\partial x}\,\vec{g}\,\vec{I}_N^T\,\tilde{t}_{fm}\,\tilde{P}_1\,\tilde{\lambda}_1\,h_1 + \frac{\partial\vec{\gamma}}{\partial y}\frac{\partial\vec{\gamma}^T}{\partial y}\,\vec{g}\,\vec{I}_N^T\,\tilde{t}_{fm}\,\tilde{P}_2\,\tilde{\lambda}_2\,h_2$$

$$+ (\frac{\partial\vec{\gamma}}{\partial y}\frac{\partial\vec{\gamma}^T}{\partial x} + \frac{\partial\vec{\gamma}}{\partial x}\frac{\partial\vec{\gamma}^T}{\partial y})\,\vec{g}\,\vec{I}_N^T\,\tilde{t}_{fm}\,\tilde{P}_3\,\tilde{\lambda}_3\,\vec{h}_3\}\,dxdy$$

$$+ \tilde{\phi}_7\,\vec{a} + \tilde{\phi}_8\,\vec{b} + \frac{\partial\vec{\gamma}}{\partial y}\,\vec{h}_2^T\,\tilde{\lambda}_2^T\,\tilde{P}_2^T\,\tilde{t}_{fm}\,\tilde{Q}_y\,\tilde{\beta}\,\vec{f}\,\frac{1}{R}$$

$$+ \frac{\partial\vec{\gamma}}{\partial x}\,\vec{h}_3^T\,\tilde{\lambda}_3^T\,\tilde{P}_3^T\,\tilde{t}_{fm}\,\tilde{Q}_y\,\tilde{\beta}\,\vec{f}\,\frac{1}{R} + \int_x\int_y \vec{\gamma}^T\,\vec{I}_N\,\vec{I}_N^T\,\tilde{t}_{fm}\,\vec{\gamma}\,\frac{\partial^2\vec{g}}{\partial t^2}\,\rho dxdy = \vec{0}$$

(B-4)

$$\tilde{\phi}_1^T\,\vec{h}_1 + \tilde{\phi}_3^T\,\vec{h}_3 + \tilde{\phi}_5\,\vec{a} + \int_x\int_y \tilde{\alpha}^T\,\tilde{Q}_x^T\,\tilde{t}_{fm}\,\tilde{Q}_x\,\tilde{\alpha}\,\frac{\partial^2\vec{e}}{\partial t^2}\,\rho dxdy = \vec{0} \qquad \text{(B-5)}$$

$$\tilde{\phi}_2^T\,\vec{h}_2 + \tilde{\phi}_4^T\,\vec{h}_3 + \tilde{\phi}_6\,\vec{b} + \tilde{\beta}^T\,\tilde{Q}_y^T\,\tilde{t}_{fm}\,\tilde{P}_2\,\tilde{\lambda}_2\,\vec{H}_2\,\frac{\partial\vec{\gamma}^T}{\partial y}\,\vec{g}\,\frac{1}{R}$$

$$+ \tilde{\beta}^T\,\tilde{Q}_y^T\,\tilde{t}_{fm}\,\tilde{P}_2\,\tilde{\gamma}_2\,\vec{h}_2\,\vec{I}_N^T\,\tilde{Q}_y\,\tilde{\beta}\,\vec{f}\,\frac{1}{R^2} + \tilde{\beta}^T\,\tilde{Q}_y^T\,\tilde{t}_{fm}\,\tilde{P}_3\,\tilde{\lambda}_3\,\vec{h}_3\,\frac{\partial\vec{\gamma}^T}{\partial x}\,\vec{g}\,\frac{1}{R}$$

$$+ \int_x\int_y \tilde{\beta}^T\,\tilde{Q}_y^T\,\tilde{t}_{fm}\,\tilde{Q}_y\,\tilde{\beta}\,\frac{\partial^2\vec{f}}{\partial t^2}\,\rho dxdy = \vec{0} \qquad \text{(B-6)}$$

$$\tilde{\phi}_7^T \vec{g} + \tilde{\phi}_5^T \vec{e} + \tilde{\phi}_9 \vec{a} - 2 \int_x \int_y \sum_{k=1}^{(N-1)} \{p_4$$

$$\cdot (\vec{a}^T \tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{R}_x \vec{\varepsilon} \vec{a} + \vec{b}^T \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{R}_y \tilde{\eta} \vec{b})^{(p_4-1)}$$

$$\cdot \tilde{\varepsilon} \tilde{R}_x^T \tilde{B}_k \tilde{t}_m \tilde{R}_x \tilde{\varepsilon} \vec{a}\} \, dxdy = \vec{0} \quad \text{(B-7)}$$

$$\tilde{\phi}_8^T \vec{g} + \tilde{\phi}_6^T \vec{f} + \tilde{\phi}_{10} \vec{b} - 2 \int_x \int_y \sum_{k=1}^{(N-1)} \{p_4$$

$$\cdot (\vec{a}^T \tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{R}_x \tilde{\varepsilon} \vec{a} + \vec{b}^T \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{R}_y \tilde{\eta} \vec{b})^{(p_4-1)}$$

$$\cdot \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{t}_m \tilde{R}_y \tilde{\eta} \vec{b}\} \, dxdy = \vec{0} \quad \text{(B-8)}$$

where

$$\tilde{\phi}_{1L} = - \int_x \int_y \tilde{\lambda}_1^T \tilde{P}_1^T \tilde{t}_{fm} [\tilde{C}_{L1} + \tilde{C}_{1L}] \tilde{P}_L \tilde{\lambda}_L \, dxdy \quad \text{(B-9)}$$

$$\tilde{\phi}_{2L} = - \int_x \int_y \tilde{\lambda}_2^T \tilde{P}_2^T \tilde{t}_{fm} [\tilde{C}_{L2} + \tilde{C}_{2L}] \tilde{P}_L \tilde{\lambda}_L \, dxdy \quad \text{(B-10)}$$

$$\tilde{\phi}_{3L} = - \int_x \int_y \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm} [\tilde{C}_{L3} + \tilde{C}_{3L}] \tilde{P}_L \tilde{\lambda}_L \, dxdy \quad \text{(B-11)}$$

$$\tilde{\phi}_1 = \int_x \int_y \tilde{\lambda}_1^T \tilde{P}_1^T \tilde{t}_{fm} \tilde{Q}_x \frac{\partial \tilde{\alpha}}{\partial x} \, dxdy \tag{B-12}$$

$$\tilde{\phi}_2 = \int_x \int_y \tilde{\lambda}_2^T \tilde{P}_2^T \tilde{t}_{fm} \tilde{Q}_y \frac{\partial \tilde{\beta}}{\partial y} \, dxdy \tag{B-13}$$

$$\tilde{\phi}_3 = \int_x \int_y \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm} \tilde{Q}_x \frac{\partial \tilde{\alpha}}{\partial y} \, dxdy \tag{B-14}$$

$$\tilde{\phi}_4 = \int_x \int_y \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm} \tilde{Q}_y \frac{\partial \tilde{\beta}}{\partial x} \, dxdy \tag{B-15}$$

$$\tilde{\phi}_5 = \int_x \int_y \tilde{\alpha}^T \tilde{Q}_x^T \overline{B}^T \tilde{t}_m \tilde{R}_x \tilde{\varepsilon} \, dxdy \tag{B-16}$$

$$\tilde{\phi}_6 = \int_x \int_y \tilde{\beta}^T \tilde{Q}_y^T \overline{B}^T \tilde{t}_m \tilde{R}_y \tilde{\eta} \, dxdy \tag{B-17}$$

$$\tilde{\phi}_7 = \int_x \int_y \frac{\partial \vec{\gamma}}{\partial x} \vec{I}^T_{(N-1)} \tilde{t}_m \tilde{R}_x \tilde{\varepsilon} \, dxdy \tag{B-18}$$

$$\tilde{\phi}_8 = \int_x \int_y \frac{\partial \vec{\gamma}}{\partial y} \vec{I}^T_{(N-1)} \tilde{t}_m \tilde{R}_y \tilde{\eta} \, dxdy \tag{B-19}$$

$$\tilde{\phi}_9 = -2 \int_x \int_y \tilde{\varepsilon}^T \tilde{R}_x^T \bar{G}_m \tilde{t}_m \tilde{R}_x \tilde{\varepsilon} \, dxdy \qquad \text{(B-20)}$$

$$\tilde{\phi}_{10} = -2 \int_x \int_y \tilde{\eta}^T \tilde{R}_y^T \bar{G}_m \tilde{t}_m \tilde{R}_y \tilde{\eta} \, dxdy \qquad \text{(B-21)}$$

and

$$\vec{B} = - \int_x \int_y \tilde{\lambda}_1^T \tilde{P}_1^T \tilde{t}_{fm} \, \delta \, \vec{I}_N \, dxdy \qquad \text{(B-22)}$$

$$\vec{G}_p = - \int_x \int_y \{ \sum_{L=1}^{3} \vec{h}_p^T \tilde{\lambda}_p^T \tilde{P}_p^T \sum_{k=1}^{N} \sum_{i=1}^{3} \sum_{j=1}^{3} (2\delta_{ij}-1)$$

$$\cdot \left[\sum_{\ell=1}^{3} \sum_{m=1}^{3} \vec{h}_\ell^T \tilde{\lambda}_\ell^T \tilde{P}_\ell^T \tilde{D}_{\ell m i j_k} \tilde{P}_m \tilde{\lambda}_m \vec{h}_m \right]^{p_{i_k}}$$

$$\cdot (\vec{T}_{jL_k} \vec{T}_{ip_k}^T + \vec{T}_{iL_K} \vec{T}_{jp_k}^T) \tilde{P}_L \tilde{\lambda}_L \vec{h}_L$$

$$+ \sum_{k=1}^{N} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{m=1}^{3} \vec{h}_p^T \tilde{\lambda}_p^T \tilde{P}_p^T [\tilde{D}_{pmij_k} + \tilde{D}_{mpij_k}^T] \tilde{P}_m \tilde{\lambda}_m \vec{h}_m$$

$$\cdot \sum_{L=1}^{3} \sum_{r=1}^{3} \vec{h}_r^T \tilde{\lambda}_r^T \tilde{P}_r^T \vec{T}_{iL_k} (2\delta_{ij}-1) p_{i_k}$$

$$\cdot \left[\sum_{\ell=1}^{3} \sum_{s=1}^{3} \vec{h}_\ell^T \tilde{\lambda}_\ell^T \tilde{P}_\ell^T \tilde{D}_{\ell s i j_k} \tilde{P}_s \tilde{\lambda}_s \vec{h}_s \right]^{(p_{i_k}-1)}$$

$$\cdot\ \vec{T}^{T}_{jM_k}\ \tilde{t}_{fm_k}\ \tilde{P}_L\ \tilde{\lambda}_L\ \vec{h}_L\ \}dxdy \tag{B-23}$$

with

$$\vec{B}_1 = -\ \alpha_x\ \Delta\ T\ \tilde{\lambda}_1^T\ \tilde{P}_1^T\ \tilde{t}_{fm}\ \vec{I}_N \tag{B-24}$$

$$\vec{B}_2 = -\ \alpha_y\ \Delta\ T\ \tilde{\lambda}_2^T\ \tilde{P}_2^T\ \tilde{t}_{fm}\ \vec{I}_N \tag{B-25}$$

APPENDIX C

NEWTON-RAPHSON ITERATIVE SOLUTION TECHNIQUE

Application of the Newton-Raphson iterative technique provides the following equations:

$$\tilde{\psi}_1^* \, \Delta\vec{g} + \tilde{\phi}_1 \, \Delta\vec{e} + \sum_{L=1}^{3} \left(\tilde{\phi}_{1L} + \frac{\partial G^*}{\partial \vec{h}_L}\right) \Delta\vec{h}_L$$

$$= B(1) - \tilde{\theta}_1^* - \tilde{\phi}_1 \, \vec{e}* - \sum_{L=1}^{3} \tilde{\phi}_{1L} \, \vec{h}_L^* - \vec{G}_1^* + \vec{B}_1 \qquad \text{(C-1)}$$

$$\tilde{\psi}_2^* \, \Delta\vec{g} + \tilde{\phi}_2 \, \Delta\vec{f} + \sum_{L=1}^{3} \left(\tilde{\phi}_{2L} + \frac{\partial G^*}{\partial \vec{h}_L}\right) \Delta\vec{h}_L + (\tilde{\psi}_{11}^* + \tilde{\psi}_{12}^*)\Delta\vec{f}^T + \tilde{\psi}_{13}^* \, \Delta\vec{g}^T$$

$$= - \tilde{\theta}_2^* - \tilde{\phi}_2 \, f* - \sum_{L=1}^{3} \tilde{\phi}_{2L} \, \vec{h}_L^* - \vec{G}_2^* + \vec{B}_2 - \vec{\theta}_{11}^* - \vec{\theta}_{12}^* \qquad \text{(C-2)}$$

$$\tilde{\psi}_3^* \, \Delta\vec{g} + \tilde{\phi}_3 \, \Delta\vec{e} + \tilde{\phi}_4 \, \Delta\vec{f} + \sum_{L=1}^{3} \left(\tilde{\phi}_{3L} + \frac{\partial G^*}{\partial \vec{h}_L}\right) \Delta\vec{h}_L + \tilde{\psi}_{14}^* \, \Delta\vec{f} + \tilde{\psi}_{15}^* \, \Delta\vec{g}$$

$$= - \tilde{\theta}_3^* - \tilde{\phi}_3 \, \vec{e}* - \tilde{\phi}_4 \, \vec{f}* - \sum_{L=1}^{3} \tilde{\phi}_{3L} \, \vec{h}_L^* - \vec{G}_3 - \vec{\theta}_{13}^* \qquad \text{(C-3)}$$

$$\tilde{\psi}_1^{*T} \, \Delta\vec{h}_1 + \tilde{\psi}_2^{*T} \, \Delta\vec{h}_2 + \tilde{\psi}_3^{*T} \, \Delta\vec{h}_3 + (\tilde{\psi}_4^* + \tilde{\psi}_5^* + \tilde{\psi}_6^*) \, \Delta\vec{g}$$

$$+ \tilde{\phi}_7 \, \Delta\vec{a} + \tilde{\phi}_8 \, \Delta\vec{b} + \tilde{\psi}_{13}^{*T} \, \Delta\vec{h}_2 + \tilde{\psi}_{15}^{*T} \, \Delta\vec{h}_3 + (\tilde{\psi}_{17}^{*T} + \tilde{\psi}_{18}^{*T})\Delta\vec{f}$$

$$= - \tilde{\theta}_4^* - \tilde{\theta}_5^* - \tilde{\theta}_6^* - \tilde{\phi}_7 \vec{a}^* - \tilde{\phi}_8 \vec{b}^* - \vec{\theta}_{17}^* - \vec{\theta}_{18}^* - \vec{\theta}_{21}^* \qquad \text{(C-4)}$$

$$\tilde{\phi}_1^T \Delta\vec{h}_1 + \tilde{\phi}_3^T \Delta\vec{h}_3 + \tilde{\phi}_5 \Delta\vec{a} = - \tilde{\phi}_1^T \vec{h}_1^* - \tilde{\phi}_3^T \vec{h}_3^* - \tilde{\phi}_5 \vec{a}^* - \vec{\theta}_{19}^* \qquad \text{(C-5)}$$

$$\tilde{\phi}_2^T \Delta\vec{h}_2 + \tilde{\phi}_4^T \Delta\vec{h}_3 + \tilde{\phi}_6 \Delta\vec{b} \; (\tilde{\psi}_{11}^{*T} + \tilde{\psi}_{12}^{*T})\Delta\vec{h}_2 + \tilde{\psi}_{14}^{*T} \Delta\vec{h}_3 + \tilde{\psi}_{16}^{*T} \Delta\vec{f} \qquad \text{(C-6)}$$

$$+ (\tilde{\psi}_{17}^{*T} + \tilde{\psi}_{18}^{*T})\Delta\vec{g} = - \tilde{\phi}_2^T \vec{h}_2^* - \tilde{\phi}_4^T \vec{h}_3^* - \tilde{\phi}_6 \vec{b}^*$$

$$- \vec{\theta}_{14}^* - \vec{\theta}_{15}^* - \vec{\theta}_{16}^* - \vec{\theta}_{20}^*$$

$$\tilde{\phi}_7^T \Delta\vec{g} + \tilde{\phi}_5^T \Delta\vec{e} + (\tilde{\phi}_9 + \tilde{\psi}_7^*) \Delta\vec{a} + \tilde{\psi}_9^* \Delta\vec{b}$$

$$= - \tilde{\phi}_7^T \vec{g}^* - \tilde{\phi}_5^T \vec{e}^* - \tilde{\phi}_9 \vec{a}^* - \tilde{\theta}_7^* \qquad \text{(C-7)}$$

$$\tilde{\phi}_8^T \Delta\vec{g} + \tilde{\phi}_6^T \Delta\vec{f} + (\tilde{\phi}_{10} + \tilde{\psi}_8^*) \Delta\vec{b} + \tilde{\psi}_{10}^* \Delta\vec{a}$$

$$= - \tilde{\phi}_8^* \vec{g}^* - \tilde{\phi}_6^T \vec{f}^* - \tilde{\phi}_{10} \vec{b}^* - \tilde{\theta}_8^* \qquad \text{(C-8)}$$

where

$$\tilde{\psi}_1^* = \int_x \int_y \tilde{\lambda}_1^T \tilde{P}_1^T \vec{I}_N \vec{g}^{*T} \tilde{t}_{fm} \frac{\partial \vec{Y}}{\partial x} \frac{\partial \vec{Y}^T}{\partial x} dxdy \tag{C-9}$$

$$\tilde{\psi}_2^* = \int_x \int_y \tilde{\lambda}_2^T \tilde{P}_2^T \vec{I}_N \vec{g}^{*T} \tilde{t}_{fm} \frac{\partial \vec{Y}}{\partial y} \frac{\partial \vec{Y}^T}{\partial y} dxdy \tag{C-10}$$

$$\tilde{\psi}_3^* = \int_x \int_y \tilde{\lambda}_3^T \tilde{P}_3^T \vec{I}_N \vec{g}^{*T} \tilde{t}_{fm} \left(\frac{\partial \vec{Y}}{\partial x} \frac{\partial \vec{Y}^T}{\partial y} + \frac{\partial \vec{Y}}{\partial y} \frac{\partial \vec{Y}^T}{\partial x} \right) dxdy \tag{C-11}$$

$$\tilde{\psi}_4^* = \int_x \int_y (\vec{I}_N^T \tilde{P}_1 \tilde{\lambda}_1 \vec{h}_1^*) \frac{\partial \vec{Y}}{\partial x} \frac{\partial \vec{Y}^T}{\partial x} \tilde{t}_{fm} dxdy \tag{C-12}$$

$$\tilde{\psi}_5^* = \int_x \int_y (\vec{I}_N^T \tilde{P}_2 \tilde{\lambda}_2 \vec{h}_2^*) \frac{\partial \vec{Y}}{\partial y} \frac{\partial \vec{Y}^T}{\partial y} \tilde{t}_{fm} dxdy \tag{C-13}$$

$$\tilde{\psi}_6^* = \int_x \int_y (\vec{I}_N^T \tilde{P}_3 \tilde{\lambda}_3 \vec{h}_3^*) \left(\frac{\partial \vec{Y}}{\partial y} \frac{\partial \vec{Y}^T}{\partial x} + \frac{\partial \vec{Y}}{\partial x} \frac{\partial \vec{Y}^T}{\partial y} \right) \tilde{t}_{fm} dxdy \tag{C-14}$$

$$\tilde{\psi}_7^* = -2 \int_x \int_y \sum_{k=1}^{(N-1)} p_4 \{ (\vec{a}^{*T} \tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{t}_m \tilde{R}_x \tilde{\varepsilon} \vec{a}^*$$

$$+ \vec{b}^{*T} \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{t}_m \tilde{R}_y \tilde{\eta} \vec{b}^*)^{(p_4 - 1)}$$

$$\cdot\ (\tilde{\varepsilon}^T\ \tilde{R}_x^T\ \tilde{B}_k\ \tilde{R}_x\ \tilde{\varepsilon}) + 2(p_4-1)$$

$$\cdot\ (\tilde{\vec{a}}*^T\ \tilde{\varepsilon}^T\ \tilde{R}_x^T\ \tilde{B}_k\ \tilde{t}_m\ \tilde{R}_x\ \tilde{\varepsilon}\ \vec{a}*$$

$$+\ \vec{b}*^T\ \tilde{\eta}^T\ \tilde{R}_y^T\ \tilde{B}_k\ \tilde{t}_m\ \tilde{R}_y\ \tilde{\eta}\ \vec{b}*)^{(p_4-2)}$$

$$\cdot\ (\tilde{\varepsilon}^T\ \tilde{R}_x^T\ \tilde{B}_k\ \tilde{R}_x\ \tilde{\varepsilon}\ \vec{a}*)$$

$$\cdot\ (\vec{a}*^T\ \tilde{\varepsilon}^T\ \tilde{R}_x^T\ \tilde{B}_k\ \tilde{R}_x\ \tilde{\varepsilon})\}\ dxdy \tag{C-15}$$

$$\tilde{\psi}_8^* = 2\int_x\int_y \sum_{k=1}^{(N-1)} p_4\{(\vec{a}*^T\ \tilde{\varepsilon}^T\ \tilde{R}_x^T\ \tilde{B}_k\ \tilde{t}_m\ \tilde{R}_x\ \tilde{\varepsilon}\ \vec{a}*$$

$$+\ \vec{b}*^T\ \tilde{\eta}^T\ \tilde{R}_y^T\ \tilde{B}_k\ \tilde{t}_m\ \tilde{R}_y\ \tilde{\eta}\ \vec{b}*)^{(p_4-1)}$$

$$\cdot\ (\tilde{\eta}^T\ \tilde{R}_y^T\ \tilde{B}_k\ \tilde{R}_y\ \tilde{\eta}) + 2(p_4-1)$$

$$\cdot\ (\vec{a}*^T\ \tilde{\varepsilon}^T\ \tilde{R}_x^T\ \tilde{B}_k\ \tilde{t}_m\ \tilde{R}_x\ \tilde{\varepsilon}\ \vec{a}*$$

$$+ \vec{b}*^T \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{t}_m \tilde{R}_y \tilde{\eta} \vec{b}*)^{(p_4-2)}$$

$$\cdot (\tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{R}_y \tilde{\eta} \vec{b}*)$$

$$\cdot (\vec{b}*^T \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{R}_y \tilde{\eta})\}dxdy \qquad \text{(C-16)}$$

$$\tilde{\psi}_9^* = -4 \int_x \int_y \sum_{k=1}^{(N-1)} p_4(p_4-1) (\vec{a}*^T \tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{t}_m \tilde{R}_x \tilde{\varepsilon} \vec{a}*$$

$$+ \vec{b}*^T \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{t}_m \tilde{R}_y \tilde{\eta} \vec{b}*)^{(p_4-2)}$$

$$\cdot (\tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{R}_x \tilde{\varepsilon} \vec{a}*) (\vec{b}*^T \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{R}_y \tilde{\eta})dxdy \qquad \text{(C-17)}$$

$$\tilde{\psi}_{10}^* = -4 \int_x \int_y p_4(p_4-1) (\vec{a}*^T \tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{t}_m \tilde{R}_x \tilde{\varepsilon} \vec{a}*$$

$$+ \vec{b}*^T \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{t}_m \tilde{R}_y \tilde{\eta} \vec{b}*)^{(p_4-2)}$$

$$\cdot (\tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{R}_y \tilde{\eta} \vec{b}*) (\vec{a}*^T \tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{R}_x \tilde{\varepsilon})dxdy \qquad \text{(C-18)}$$

$$\tilde{\psi}^*_{11} = \iiint_V \left(\frac{1}{R} \tilde{\lambda}_2^T \tilde{P}_2^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta} \frac{\partial \vec{\gamma}^T}{\partial y} \vec{g}^*\right) dV ,$$

$$\tilde{\psi}^*_{12} = \iiint_V \left(\frac{2}{R^2} \tilde{\lambda}_2^T \tilde{P}_2^T \vec{I}_N \vec{f}^{T*} \tilde{\beta}^T \tilde{Q}_y^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta}\right) dV ,$$

$$\tilde{\psi}^*_{13} = \iiint_V \left(\frac{1}{R} \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta} \frac{\partial \vec{\gamma}^T}{\partial x} \vec{g}^*\right) dV ,$$

$$\tilde{\psi}^*_{14} = \iiint_V \left(\frac{1}{R} \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta} \vec{f}^* \frac{\partial \vec{\gamma}^T}{\partial x}\right) dV ,$$

$$\tilde{\psi}^*_{15} = \iiint_V \left(\frac{1}{R} \tilde{\lambda}_3^T \tilde{P}_3^T \tilde{t}_{fm} \tilde{Q}_y \tilde{\beta} \vec{f}^* \frac{\partial \vec{\gamma}^T}{\partial x}\right) dV ,$$

$$\tilde{\psi}^*_{16} = \left(\frac{1}{R^2} \tilde{\beta}^T \tilde{Q}_y^T \tilde{t}_{fm} \tilde{P}_2 \tilde{\lambda}_2 \vec{h}_2^* \vec{I}_N^T \tilde{Q}_y \tilde{\beta}\right) dV ,$$

$$\tilde{\psi}^*_{17} = \iiint_V \left(\frac{1}{R} \tilde{\beta}^T \tilde{Q}_y^T \tilde{t}_{fm} \tilde{P}_2 \tilde{\lambda}_2 \vec{h}_2^* \frac{\partial \vec{\gamma}^T}{\partial y}\right) dV ,$$

$$\tilde{\psi}^*_{18} = \iiint_V \left(\frac{1}{R} \tilde{\beta}^T \tilde{Q}_y \tilde{t}_{fm} \tilde{P}_3 \tilde{\lambda}_3 \vec{h}_3^* \frac{\partial \vec{\gamma}^T}{\partial y}\right) dV \qquad \text{(C-19)}$$

and

$$\tilde{\theta}_1^* = \frac{1}{2} \int_x \int_y \tilde{\lambda}_1^T \tilde{P}_1^T \vec{I}_N \vec{g}^{*T} \frac{\partial \vec{Y}}{\partial x} \frac{\partial \vec{Y}^T}{\partial x} \tilde{t}_{fm} \vec{g}^* \, dxdy \tag{C-20}$$

$$\tilde{\theta}_2^* = \frac{1}{2} \int_x \int_y \tilde{\lambda}_2^T \tilde{P}_2^T \vec{I}_N \vec{g}^{*T} \frac{\partial \vec{Y}}{\partial y} \frac{\partial \vec{Y}^T}{\partial y} \tilde{t}_{fm} \vec{g}^* \, dxdy \tag{C-21}$$

$$\tilde{\theta}_2^* = \frac{1}{2} \int_x \int_y \tilde{\lambda}_3^T \tilde{P}_3^T \vec{I}_N \vec{g}^{*T} \frac{\partial \vec{Y}}{\partial x} \frac{\partial \vec{Y}^T}{\partial y} \tilde{t}_{fm} \vec{g}^* \, dxdy \tag{C-22}$$

$$\tilde{\theta}_4^* = \int_x \int_y \frac{\partial \vec{Y}}{\partial x} \frac{\partial \vec{Y}^T}{\partial x} \tilde{t}_{fm} \vec{g}^* \vec{I}_N^T \tilde{P}_1 \tilde{\lambda}_1 \vec{h}_1^* \, dxdy \tag{C-23}$$

$$\tilde{\theta}_5^* = \int_x \int_y \frac{\partial \vec{Y}}{\partial y} \frac{\partial \vec{Y}^T}{\partial y} \tilde{t}_{fm} \vec{g}^* \vec{I}_N^T \tilde{P}_2 \tilde{\lambda}_2 \vec{h}_2^* \, dxdy \tag{C-24}$$

$$\tilde{\theta}_6^* = \int_x \int_y \left(\frac{\partial \vec{Y}}{\partial y} \frac{\partial \vec{Y}^T}{\partial x} + \frac{\partial \vec{Y}}{\partial x} \frac{\partial \vec{Y}^T}{\partial y} \right) \tilde{t}_{fm} \vec{g}^* \vec{I}_N^T \tilde{P}_3 \tilde{\lambda}_3 \vec{h}_3^* \, dxdy \tag{C-25}$$

$$\tilde{\theta}_7^* = 2 \int_x \int_y \sum_{k=1}^{(N-1)} p_4 (\vec{a}^{*T} \tilde{\varepsilon}^T \tilde{R}_x^T \tilde{B}_k \tilde{t}_m \tilde{R}_x \tilde{\varepsilon} \vec{a}^*$$

$$+ \vec{b}^{*T} \tilde{\eta}^T \tilde{R}_y^T \tilde{B}_k \tilde{t}_m \tilde{R}_y \tilde{\eta} \vec{b}^*)^{(p_4 - 1)}$$

$$\cdot\ (\tilde{\varepsilon}^T\ \tilde{R}_x^T\ \tilde{B}_k\ \tilde{R}_x\ \tilde{\varepsilon}\ \vec{a}*)\ dxdy \qquad (C\text{-}26)$$

$$\tilde{\theta}^*_8 = 2 \int_x \int_y \sum_{k=1}^{(N-1)} p_4 (\vec{a}*^T\ \tilde{\varepsilon}^T\ \tilde{R}_x^T\ \tilde{B}_k\ \tilde{t}_m\ \tilde{R}_x\ \tilde{\varepsilon}\ \vec{a}*$$

$$+\ \vec{b}*^T\ \tilde{\eta}^T\ \tilde{R}_y^T\ \tilde{B}_k\ \tilde{t}_m\ \tilde{R}_y\ \tilde{\eta}\ \vec{b}*)^{(p_4-1)}$$

$$\cdot\ (\tilde{\eta}^T\ \tilde{R}_y^T\ \tilde{B}_k\ \tilde{R}_y\ \tilde{\eta}\ \vec{b}*)\ dxdy \qquad (C\text{-}27)$$

$$\vec{\theta}^*_{11} = \iiint_V (\frac{1}{R}\ \tilde{\lambda}_2^T\ \tilde{P}_2^T\ \tilde{t}_{fm}\ \tilde{Q}_y\ \tilde{\beta}\ \vec{f}*\ \frac{\partial \vec{\gamma}^T}{\partial y}\ \vec{g}*) dV\ ,$$

$$\vec{\theta}^*_{12} = \iiint_V (\frac{1}{R^2}\ \tilde{\lambda}_2^T\ \tilde{P}_2^T\ \vec{I}_N\ \vec{f}^{T*}\ \tilde{\beta}^T\ \tilde{Q}_y^T\ \tilde{t}_{fm}\ \tilde{Q}_y\ \tilde{\beta}\ \vec{f}*) dV\ ,$$

$$\vec{\theta}^*_{13} = \iiint_V (\frac{1}{R}\ \tilde{\lambda}_3^T\ \tilde{P}_3^T\ \tilde{t}_{fm}\ \tilde{Q}_y\ \tilde{\beta}\ \vec{f}*\ \frac{\partial \vec{\gamma}^T}{\partial x}\ \vec{g}*) dV\ ,$$

$$\vec{\theta}^*_{14} = \iiint_V (\frac{1}{R}\ \tilde{\beta}^T\ \tilde{Q}_y^T\ \tilde{t}_{fm}\ \tilde{P}_2\ \tilde{\lambda}_2\ \vec{h}^*_2\ \frac{\partial \vec{\gamma}^T}{\partial y}\ \vec{g}*) dV\ ,$$

$$\vec{\theta}^*_{15} = \iiint_V (\frac{1}{R^2}\ \tilde{\beta}^T\ \tilde{Q}_y^T\ \tilde{t}_{fm}\ \tilde{P}_2\ \tilde{\lambda}_2\ \vec{h}^*_2\ \vec{I}_N^T\ \tilde{Q}_y\ \tilde{\beta}\ \vec{f}*) dV\ ,$$

$$\vec{\theta}^{*}_{16} = \iiint_V \left(\frac{1}{R}\, \tilde{\beta}^T\, \tilde{Q}_y^T\, \tilde{t}_{fm}\, \tilde{P}_3\, \tilde{\lambda}_3\, \vec{h}^{*}_3\, \frac{\partial \vec{\gamma}^T}{\partial x}\, \vec{g}^{*}\right) dV ,$$

$$\vec{\theta}^{*}_{17} = \iiint_V \left(\frac{1}{R}\, \frac{\partial \vec{\gamma}}{\partial y}\, \vec{h}_2^{T*}\, \tilde{\lambda}_2^T\, \tilde{P}_2^T\, \tilde{t}_{fm}\, \tilde{Q}_y\, \tilde{\beta}\, \vec{f}^{*}\right) dV ,$$

$$\vec{\theta}^{*}_{18} = \iiint_V \left(\frac{1}{R}\, \frac{\partial \vec{\gamma}}{\partial x}\, \vec{h}_3^{T*}\, \tilde{\lambda}_3^T\, \tilde{P}_3^T\, \tilde{t}_{fm}\, \tilde{Q}_y\, \tilde{\beta}\, \vec{f}^{*}\right) dV ,$$

$$\vec{\theta}^{*}_{19} = \iiint_V \left(\tilde{\alpha}^T\, \tilde{Q}_x^T\, \tilde{t}_{fm}\, \tilde{Q}_x\, \tilde{\alpha}\right) dV ,$$

$$\vec{\theta}^{*}_{20} = \iiint_V \left(\tilde{\beta}^T\, \tilde{Q}_y^T\, \tilde{t}_{fm}\, \tilde{Q}_y\, \tilde{\beta}\right) dV ,$$

$$\vec{\theta}^{*}_{21} = \iiint_V \left(\tilde{\gamma}^T\, \vec{I}_N\, \vec{I}_N^T\, \tilde{t}_{fm}\, \tilde{\gamma}\right) dV$$

$$\frac{\partial \vec{G}_p}{\partial \vec{h}_q} = - \int_x \int_y \sum_{k=1}^{N} \sum_{i=1}^{3} \sum_{j=1}^{3} 2(\delta_{ij} - 1)$$

$$\cdot \left[\sum_{\ell=1}^{3} \sum_{m=1}^{3} \vec{h}_\ell^T\, \tilde{\lambda}_\ell^T\, \tilde{P}_\ell^T\, \tilde{D}_{\ell m i j_k}\, \tilde{P}_m\, \tilde{\lambda}_m\, \vec{h}_m \right]^{p_{i_k}}$$

$$\cdot\ \tilde{\lambda}_p^T\, \tilde{P}_p^T\, (\vec{T}_{jq_k}\, \vec{T}^T_{ip_k} + \vec{T}_{iq_k}\, \vec{T}^T_{jp_k})\, \tilde{t}_{fm_k}\, \tilde{P}_q\, \tilde{\lambda}_q$$

$$+ \; p_{i_k} \left[\sum_{\ell=1}^{3} \sum_{m=1}^{3} \vec{h}_\ell^T \, \tilde{\lambda}_\ell^T \, \tilde{P}_\ell^T \, \tilde{D}_{\ell m i j_k} \, \tilde{P}_m \, \tilde{\lambda}_m \, \vec{h}_m \right]^{(p_{i_k}-1)}$$

$$\cdot \; \sum_{L=1}^{3} \tilde{\lambda}_p^T \, \tilde{P}_p^T \, (\vec{T}_{jL_k} \, \vec{T}_{ip_k}^T + \vec{T}_{iL_k} \, \vec{T}_{jp_k}^T) \, \tilde{P}_L \, \tilde{\lambda}_L \, \vec{h}_L$$

$$\cdot \; \sum_{\ell=1}^{3} \vec{h}_\ell^T \, \tilde{\lambda}_\ell^T \, \tilde{P}_\ell^T \, (\tilde{D}_{\ell q i j_k} + \tilde{D}_{q \ell i j_k}) \, \tilde{P}_q \, \tilde{\lambda}_q$$

$$+ \; \sum_{L=1}^{3} \sum_{M=1}^{3} \vec{h}_M^T \, \tilde{\lambda}_M^T \, \tilde{P}_M^T \, \vec{T}_{iL_k} \, \vec{T}_{jM_k}^T \, \tilde{t}_{fm_k} \, \tilde{P}_L \, \tilde{\lambda}_L \, \vec{h}_L$$

$$\cdot \; \tilde{\lambda}_p^T \, \tilde{P}_p^T \, (\tilde{D}_{pmij_k} + \tilde{D}_{mpij_k}) \, \tilde{P}_q \, \tilde{\lambda}_q$$

$$+ \; \sum_{M=1}^{3} \tilde{\lambda}_p^T \, \tilde{P}_p^T \, (\tilde{D}_{pmij_k} + \tilde{D}_{mpij_k}) \, \tilde{P}_m \, \tilde{\lambda}_m \, \vec{h}_m$$

$$\cdot \; \sum_{L=1}^{3} \vec{h}_L^T \, \tilde{\lambda}_L^T \, \tilde{P}_L^T \, (\vec{T}_{iq_k} \, \vec{T}_{jL_k}^T + \vec{T}_{jq_k} \, \vec{T}_{iL_k}^T) \, \tilde{t}_{fm_k} \, \tilde{P}_q \, \tilde{\lambda}_q$$

$$+ \; p_{i_k} \, (p_{i_k}-1) \left[\sum_{\ell=1}^{3} \sum_{m=1}^{3} \vec{h}_\ell^T \, \tilde{\lambda}_\ell^T \, \tilde{P}_\ell^T \, \tilde{D}_{\ell m i j_k} \, \tilde{P}_m \, \tilde{\lambda}_m \, \vec{h}_m \right]^{(p_{i_k}-2)}$$

$$\cdot \sum_{L=1}^{3} \sum_{M=1}^{3} \vec{h}_M^T \tilde{\lambda}_M^T \tilde{P}_M^T \vec{T}_{iL_k} \vec{T}_{jM_k}^T \tilde{t}_{fm_k} \tilde{P}_L \tilde{\lambda}_L \vec{h}_L$$

$$\cdot \sum_{m=1}^{3} \tilde{\lambda}_p^T \tilde{P}_p^T (\tilde{D}_{pmij_k} + \tilde{D}_{mpij_k}) \tilde{P}_m \tilde{\lambda}_m \vec{h}_m$$

$$\cdot \sum_{\ell=1}^{3} \vec{h}_\ell^T \tilde{\lambda}_\ell^T \tilde{P}_\ell^T (\tilde{D}_{\ell qij_k} + \tilde{D}_{q\ell ij_k}) \tilde{P}_q \tilde{\lambda}_q \, dxdy \qquad \text{(C-28)}$$

APPENDIX D

GOVERNING EQUATIONS IN VECTOR-MATRIX FORM

The terms of Equation (35) are defined as follows:

$$
\tilde{A}^* = \begin{array}{c}
\begin{array}{cccccccccc}
\Delta\vec{h}_1 & \Delta\vec{h}_2 & \Delta\vec{h}_3 & \Delta\vec{g} & \Delta\vec{e} & \Delta\vec{f} & \Delta\vec{a} & \Delta\vec{b} & \Delta\vec{h}_4 & \Delta\vec{h}_6
\end{array} \\
\left[\begin{array}{cccccccccc}
\tilde{\phi}_{11} + \frac{\partial\vec{G}_1^*}{\partial\vec{h}_1} & \tilde{\phi}_{12} + \frac{\partial\vec{G}_1^*}{\partial\vec{h}_2} & \tilde{\phi}_{13} + \frac{\partial\vec{G}_1^*}{\partial\vec{h}_3} & \tilde{\psi}_1^* & \tilde{\phi}_1 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\phi}_{21} + \frac{\partial\vec{G}_2^*}{\partial\vec{h}_1} & \tilde{\phi}_{22} + \frac{\partial\vec{G}_2^*}{\partial\vec{h}_2} & \tilde{\phi}_{23} + \frac{\partial\vec{G}_2^*}{\partial\vec{h}_3} & \tilde{\psi}_2^* + \tilde{\phi}_{24} + \tilde{\psi}_{13}^* & 0 & \tilde{\phi}_2 + \tilde{\psi}_{11}^* + \tilde{\psi}_{12}^* & 0 & 0 & 0 & 0 \\
\tilde{\phi}_{31} + \frac{\partial\vec{G}_3^*}{\partial\vec{h}_1} & \tilde{\phi}_{32} + \frac{\partial\vec{G}_3^*}{\partial\vec{h}_2} & \tilde{\phi}_{33} + \frac{\partial\vec{G}_3^*}{\partial\vec{h}_3} & \tilde{\psi}_3^* + \tilde{\psi}_{15}^* & \tilde{\phi}_3 & \tilde{\phi}_4 + \tilde{\psi}_{14}^* & 0 & 0 & 0 & 0 \\
\tilde{\psi}_1^{*T} & \tilde{\psi}_2^{*T} + \tilde{\phi}_{24}^T + \tilde{\psi}_{13}^{*T} & \tilde{\psi}_3^{*T} + \tilde{\psi}_{15}^{*T} & \tilde{\psi}_4^* + \tilde{\psi}_5^* + \tilde{\psi}_6^* \; \tilde{\psi}_{24}^* & 0 & \tilde{\psi}_{17}^* + \tilde{\psi}_{18}^* & \tilde{\phi}_7 & \tilde{\phi}_8 & \tilde{\psi}_{14}^* & \tilde{\phi}_{16} \\
\tilde{\phi}_1^T & 0 & \tilde{\phi}_3^T & 0 & 0 & 0 & \tilde{\phi}_5 & 0 & \tilde{\phi}_{14} & 0 \\
0 & \tilde{\phi}_2^T + \tilde{\psi}_{11}^{*T} + \tilde{\psi}^{*T}_{12} & \tilde{\phi}_4^T + \tilde{\psi}_{14}^{*T} & \tilde{\psi}_{17}^{*T} + \tilde{\psi}_{18}^{*T} & 0 & \tilde{\psi}_{16}^{*T} & 0 & \tilde{\phi}_6 & 0 & 0 \\
0 & 0 & 0 & \tilde{\phi}_7^T & \tilde{\phi}_5^T & 0 & \tilde{\phi}_9 + \tilde{\psi}_7^* & \tilde{\psi}_9^* & 0 & 0 \\
0 & 0 & 0 & \tilde{\phi}_8^T & 0 & \tilde{\phi}_6^T & 0 & \tilde{\phi}_{10} + \tilde{\psi}_8^* & 0 & 0 \\
0 & 0 & 0 & \tilde{\psi}_{14}^{*T} & \tilde{\phi}_{14}^T & 0 & 0 & 0 & \tilde{\phi}_{44} + \frac{\partial\vec{Z}_4^*}{\partial\vec{h}_4} & 0 \\
0 & 0 & 0 & \tilde{\phi}_{16}^T & 0 & 0 & 0 & 0 & 0 & \tilde{\phi}_{66} + \frac{\partial\vec{Z}_6^*}{\partial\vec{h}_6}
\end{array}\right]
\end{array}
$$

$$\Delta X = \begin{bmatrix} \Delta\vec{h}_1 \\ \Delta\vec{h}_2 \\ \Delta\vec{h}_3 \\ \Delta\vec{g} \\ \Delta\vec{e} \\ \Delta\vec{f} \\ \Delta\vec{a} \\ \Delta\vec{b} \\ \Delta\vec{h}_4 \\ \Delta\vec{h}_6 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix}
-\tilde{\phi}_{11} & -\tilde{\phi}_{12} & -\tilde{\phi}_{13} & 0 & -\tilde{\phi}_{1} & 0 & 0 & 0 & 0 & 0 \\
-\tilde{\phi}^T_{12} & -\tilde{\phi}_{22} & -\tilde{\phi}_{23} & -\tilde{\phi}_{24} & 0 & -\tilde{\phi}_{2} & 0 & 0 & 0 & 0 \\
-\tilde{\phi}^T_{13} & -\tilde{\phi}^T_{23} & -\tilde{\phi}_{33} & 0 & -\tilde{\phi}_{3} & -\tilde{\phi}_{4} & 0 & 0 & 0 & 0 \\
0 & -\tilde{\phi}^T_{24} & 0 & 0 & 0 & 0 & -\tilde{\phi}_{7} & -\tilde{\phi}_{8} & 0 & -\tilde{\phi}_{16} \\
-\tilde{\phi}^T_{1} & 0 & -\tilde{\phi}^T_{3} & 0 & 0 & 0 & -\tilde{\phi}_{5} & 0 & -\tilde{\phi}_{14} & 0 \\
0 & -\tilde{\phi}^T_{2} & -\tilde{\phi}^T_{4} & 0 & 0 & 0 & 0 & -\tilde{\phi}_{6} & 0 & 0 \\
0 & 0 & 0 & -\tilde{\phi}^T_{7} & -\tilde{\phi}^T_{5} & 0 & -\tilde{\phi}_{9} & 0 & 0 & 0 \\
0 & 0 & 0 & -\tilde{\phi}^T_{8} & 0 & -\tilde{\phi}^T_{6} & 0 & -\tilde{\phi}_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & -\tilde{\phi}^T_{14} & 0 & 0 & 0 & -\tilde{\phi}_{44} & 0 \\
0 & 0 & 0 & -\tilde{\phi}^T_{16} & 0 & 0 & 0 & 0 & 0 & -\tilde{\phi}_{66}
\end{bmatrix}$$

$$\vec{X}^* = \begin{bmatrix} \vec{h}_1^* \\ \vec{h}_2^* \\ \vec{h}_3^* \\ \vec{g}^* \\ \vec{e}^* \\ \vec{f}^* \\ \vec{a}^* \\ \vec{b}^* \\ \vec{h}_4^* \\ \vec{h}_6^* \end{bmatrix}$$

$$
\begin{array}{cccc}
 & \text{LOADS} & \begin{array}{c}\text{NONLINEAR}\\ \text{MATERIALS}\end{array} & \begin{array}{c}\text{NONLINEAR}\\ \text{GEOMETRY}\end{array} \\
\vec{F}_{ext} = & \multicolumn{3}{c}{\left[\begin{array}{ccl}
\vec{B}_1 & -\vec{G}^*_1 & -\vec{\theta}^*_1 \\
 & -\vec{G}^*_2 & -\vec{\theta}^*_2 \ -\vec{\theta}^*_{11} \ -\vec{\theta}^*_{12} \\
 & -\vec{G}^*_3 & -\vec{\theta}^*_3 \ -\vec{\theta}^*_{13} \\
 & & -\vec{\theta}^*_{17} \ -\vec{\theta}^*_{18} \\
 & & -\vec{\theta}^*_4 \ -\vec{\theta}^*_5 \ -\vec{\theta}^*_6 \ -\vec{\theta}^*_{24} \\
 & & \\
 & & -\vec{\theta}^*_{14} \ -\vec{\theta}^*_{15} \ -\vec{\theta}^*_{16} \\
 & \vec{G}^*_7 & \\
 & \vec{G}^*_8 & \\
\vec{B}_s & -\vec{G}^*_4 & -\vec{\theta}^*_{14} \\
 & -\vec{G}^*_6 &
\end{array}\right]}
\end{array}
$$

THERMAL LOADS

$$\vec{B} = \begin{bmatrix} \vec{B}_{TX} \\ \vec{B}_{TY} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

DYNAMIC TERMS

$$\tilde{M} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\vec{\theta}^{*}_{21} \\ -\vec{\theta}^{*}_{19} \\ -\vec{\theta}^{*}_{20} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

DYNAMIC TERMS

$$\ddot{X}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\partial^2 \vec{g}^*}{\partial t^2} \\ \dfrac{\partial^2 \vec{e}^*}{\partial t^2} \\ \dfrac{\partial^2 \vec{f}^*}{\partial t^2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$